CURSO DE RELATIVIDAD GENERAL

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Demostrar la expresión del capítulo 57 del curso de Relatividad General, de la forma:

$$J^{\mu} = g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta} - g^{\beta\mu}\delta\Gamma^{\lambda}_{\lambda\beta} = g^{\mu\nu}g^{\alpha\beta}\left(\partial_{\alpha}\delta g_{\nu\beta} - \partial_{\nu}\delta g_{\alpha\beta}\right), \tag{1}$$

donde $g_{\alpha\beta}$, $g^{\alpha\beta}$, $\Gamma^{\alpha}_{\beta\delta}$ y $\delta\Gamma^{\alpha}_{\beta\delta}$ son el tensor métrico, la métrica inversa, los símbolos de Christoffel y la variación de los mismos, respectivamente.

En el capítulo 57 se demostraron las siguientes dos igualdades,

$$g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta} = \tfrac{1}{2}g^{\alpha\beta}g^{\mu\nu}\left(\partial_{\beta}\delta g_{\nu\alpha} + \partial_{\alpha}\delta g_{\nu\beta} - \partial_{\nu}\delta g_{\alpha\beta}\right), \tag{2}$$

$$g^{\eta\mu}\delta\Gamma^{\lambda}_{\lambda\eta} = \tfrac{1}{2}g^{\eta\mu}g^{\lambda\gamma}\left(\partial_{\eta}\delta g_{\gamma\lambda} + \partial_{\lambda}\delta g_{\gamma\eta} - \partial_{\gamma}\delta g_{\lambda\eta}\right). \tag{3}$$

Comenzamos considerando que (3) posee tres índices mudos η, γ, λ, mismos que podemos reemplazar por $\eta, \gamma, \lambda \rightarrow \nu, \alpha, \beta$, tal que (3) se escribe como,

$$g^{\nu\mu}\delta\Gamma^{\beta}_{\beta\nu} = \tfrac{1}{2}g^{\nu\mu}g^{\beta\alpha}\left(\partial_{\nu}\delta g_{\alpha\beta} + \partial_{\beta}\delta g_{\alpha\nu} - \partial_{\alpha}\delta g_{\beta\nu}\right). \tag{4}$$

Ahora procedemos a calcular la expresión,

$$J^{\mu} = g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta} - g^{\nu\mu}\delta\Gamma^{\beta}_{\beta\nu}. \tag{5}$$

Reemplazando (2) y (4) en (5), es claro que,

$$\begin{aligned} J^{\mu} &= g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta} - g^{\nu\mu}\delta\Gamma^{\beta}_{\beta\nu}, \\ &= \frac{1}{2}g^{\alpha\beta}g^{\mu\nu}\left(\partial_{\beta}\delta g_{\nu\alpha} + \partial_{\alpha}\delta g_{\nu\beta} - \partial_{\nu}\delta g_{\alpha\beta} - \partial_{\nu}\delta g_{\alpha\beta} - \partial_{\beta}\delta g_{\alpha\nu} + \partial_{\alpha}\delta g_{\beta\nu}\right). \end{aligned} \tag{6}$$

Nótese que en la expresión anterior el primer término se anula con el quinto, tal que,

$$J^{\mu} = \frac{1}{2}g^{\alpha\beta}g^{\mu\nu}\left(\partial_{\alpha}\delta g_{\nu\beta} - \partial_{\nu}\delta g_{\alpha\beta} - \partial_{\nu}\delta g_{\alpha\beta} + \partial_{\alpha}\delta g_{\beta\nu}\right). \tag{7}$$

Considerando que el tensor métrico es simétrico, $g_{\alpha\beta} = g_{\beta\alpha}$, la expresión se simplifica,

$$\begin{aligned} J^{\mu} &= \frac{1}{2}g^{\alpha\beta}g^{\mu\nu}\left(2\partial_{\alpha}\delta g_{\beta\nu} - 2\partial_{\nu}\delta g_{\alpha\beta}\right). \\ J^{\mu} &= g^{\alpha\beta}g^{\mu\nu}\left(\partial_{\alpha}\delta g_{\beta\nu} - \partial_{\nu}\delta g_{\alpha\beta}\right). \end{aligned}$$

Finalmente, obtenemos el resultado deseado,

$$J^{\mu} = g^{\alpha\beta}g^{\mu\nu}\left(\partial_{\alpha}\delta g_{\beta\nu} - \partial_{\nu}\delta g_{\alpha\beta}\right).$$